Enrollment No: ______________________ Exam Seat No:______________________

# C.U.SHAH UNIVERSITY

## Summer Examination-2017

**Subject Name : Ring Theory**

**Subject Code : 4SC06RTC1** **Branch: B. Sc.(Mathematics)**

**Semester : 6** **Date : 17/04/2017** **Time : 02:30 To 05:30** **Marks : 70**

Instructions:

(1) Use of Programmable calculator & any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

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**Q-1** **Attempt the following questions:** **(14)**

**a)** Define: Boolean ring

**b)** Find zero divisors of the ring $\left(Z_6;\ +;\ \cdot\right)$.

**c)** Find characteristic of the ring $\left(Z_5;\ +_5;\ \cdot_5\right)$.

**d)** Give an example of sub ring which is not an ideal.

**e)** Show that an integral domain contains no idempotent except 0 or 1.

**f)** Define: Quotient ring

**g)** Show that a ring R is commutative if $a^2 = a$ for each $a \in R$.

**h)** Show that the ring $\left(P(U);\ \Delta;\ \cap\right)$ is not an integral domain if U contains more than one element.

**i)** Define: Principal ideal ring

**j)** Prove that the mapping $f:\left(Z;\ +;\ \cdot\right)\to\left(Z_n;\ +_n;\ \cdot_n\right)$ where $\phi(m)=[m],\ m\in Z$ is an onto homomorphism.

**k)** If the homomorphism $\phi:\left(Z;\ +;\ \cdot\right)\to\left(Z_2;\ +_2;\ \cdot_2\right)$ is defined as

$$\phi(x)=\begin{cases}0 & ;x \text{ is even}\\ 1 & ;x \text{ is odd}\end{cases}$$

then find kernel of $\phi$.

**l)** Find zeros of $f(x)=x^2-1$ in $Z_{15}$.

**m)** Show that the polynomial $x^3+x+1$ is irreducible in $Z_5[x]$.

**n)** Define: Monic polynomial

**Attempt any four questions from Q-2 to Q-8**

**Q-2** **Attempt all questions** **(14)**

**a)** For given subrings $U_1$ and $U_2$ of a ring $R$, show that their intersection $U_1 \cap U_2$ is also a subring of $R$. **(5)**

**b)** Show that for elements $a$ and $b$ of a ring $R$, **(5)**

(i) $a0 = 0a = 0$ (ii) $a(-b) = (-a)b = -(ab)$ (iii) $(-a)(-b) = ab$

If $R$ is a ring with unity, then

(iv) $(-1)a = -a$ (v) $(-1)(-1) = 1$

**c)** Prove that a nonzero element $[m]$ of ring $(Z_n;\ +_n;\ \cdot_n)$ is a zero divisor iff $m$ and $n$ are not relatively prime. **(4)**

**Q-3** **Attempt all questions** **(14)**

**a)** Let $I = 8Z$ in the ring $R = (2Z;\ +;\ \cdot)$. Prepare addition and multiplication tables for the quotient ring $R/I$. **(5)**

**b)** If a commutative ring $R$ with unity has no proper ideal, then prove that $R$ is a field. **(5)**

**c)** Show that the characteristic of an integral domain is either a prime number or zero. **(4)**

**Q-4** **Attempt all questions** **(14)**

**a)** Prove that a non empty subset $U$ of a ring $R$ is a subring of $R$ iff the following conditions are satisfied. **(5)**

(i) $a - b \in U$ and (ii) $ab \in U$ for $a, b \in U$

**b)** Let $(M_2(Z);\ +;\ \cdot)$ be a ring. Check whether the $I = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| a, b \in Z \right\}$ is an ideal of $M_2(Z)$ or not. **(5)**

**c)** Prove that a homomorphism defined on the ring $(Z;\ +;\ \cdot)$ is either a zero homomorphism or identity mapping. **(4)**

**Q-5** **Attempt all questions** **(14)**

**a)** If $(R;\ +;\ \cdot)$ is a ring wih unity then prove that the mapping $\phi : (Z;\ +;\ \cdot) \to (R;\ +;\ \cdot)$, where $\phi(n) = n1,\ n \in Z$, is a homomorphism with **(5)**

(i) $K_\phi = \langle m \rangle$ if the characteristic of $R$ is $m$, and

(ii) $K_\phi = \{0\}$ if the characteristic of R is zero

**b)** Prove that the restricted cancellation law for multiplication holds good in a commutative ring iff it has no zero divisors. **(5)**

**c)** Obtain the quotient $q(x)$ and remainder $r(x)$ on dividing $f(x) = 3x^3 + 2x^2 + x + 1$ by $g(x) = x^2 + 3x + 2$ in $Z_5[x]$ and express $f(x)$ in the form $q(x)g(x) + r(x)$. **(4)**

**Q-6** **Attempt all questions** **(14)**

**a)** Prove that for ideals $I_1$ and $I_2$ of a ring $R$, $I_1 \cup I_2$ is also an ideal of $R$ iff either $I_1 \subset I_2$ or $I_2 \subset I_1$. **(5)**

**b)** Obtain all principal ideals in the ring $(Z_{12};\ +_{12};\ \cdot_{12})$. **(5)**

**c)** If for $f(x) = (1, -2, 0, 3, 0, ..........)$ and $g(x) = (2, 0, -3, 0, 4, 0, .........) \in Z[x]$ then find $f(x) + g(x)$ and $f(x) \cdot g(x)$. **(4)**

**Q-7** **Attempt all questions** **(14)**

**a)** Find the g.c.d. of $f(x) = 6x^3 + 5x^2 - 2x + 25$ and $g(x) = 2x^2 - 3x + 5 \in R[x]$ and **(5)**

express it in the form $a(x)f(x)+b(x)g(x)$ .

**b)** Prove that for nonzero polynomials, $f,g\in D[x]$, $[fg]=[f]+[g]$ . **(5)**

**c)** Prove that a finite integral domain is a field. **(4)**

**Q-8** **Attempt all questions** **(14)**

**a)** If we define addition and multiplication in power set $P(U)$, $U$ being the universal set, as follows: **(7)**

For $A,B\in P(U)$

$$A+B=A\Delta B=(A\cup B)-(A\cap B)$$

$$A\cdot B=A\cap B$$

then show that $(P(U);\ +;\ \cdot)$ is a ring.

**b)** Prove that intersection of ideals is an ideal in a ring. **(7)**